Javier Nogales
@fjnogales

Dantzig already said it: if we do not know how to make decisions under uncertainty, we are not planning the problem right 👇

Tweet übersetzen

"Those of us who were doing the planning right from the very beginning understood that the real problem was to be able to do planning under uncertainty. This, I feel, is the real field we should all be working on."

– G. B. Dantzig, E-Optimization (2001)

2:18 nachm. · 23. Okt. 2020 · Twitter Web App

Robust Linear Complementarity Problems

Martin Schmidt

October 26, 2020, University of Maryland/Zoom

Trier University

Overview

A Primer on Robust Optimization

Linear Complementarity Problems

(Why) Robust Linear Complementarity Problems

Γ-Robust LCPs

Adjustable Robust LCPs

Conclusion

This is joint work with

- **Vanessa Krebs**
- Christian Biefel
- Emre Çelebi
- Anja Kramer
- Frauke Liers
- Michael Müller
- Jan Rolfes

Overview

Convex Quadratic Programming

$$\min_{x \in \mathbb{R}^n} \quad \frac{1}{2} x^\top Q x + c^\top x \quad \text{s.t.} \quad Ax \leq b, \quad Cx = d$$

- $Q \in \mathbb{R}^{n \times n}$ is a symmetric and positive semi-definite matrix
- $A \in \mathbb{R}^{m \times n}$, $C \in \mathbb{R}^{k \times n}$
- $b \in \mathbb{R}^m$, $d \in \mathbb{R}^k$, $c \in \mathbb{R}^n$

Towards the Robust Counterpart

Uncertain QP data

- QP data (Q, A, C, c, b, d) are uncertain
- In particular: contained in a given uncertainty set $\mathcal{U}$

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Uncertain convex QP

$$\left\{ \min_{x \in \mathbb{R}^n} \left\{ \frac{1}{2} x^\top Q x + c^\top x \colon Ax \leq b,\ Cx = d \right\} \right\}_{(Q,A,C,c,b,d) \in \mathcal{U}}$$

- Family of optimization problems of the nominal type
- Abbreviation $u := (Q, A, C, c, b, d)$

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- Family of optimization problems of the nominal type
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Robust counterpart

$$\min_{x \in \mathbb{R}^n} \left\{ \sup_{u \in \mathcal{U}} \left\{ \frac{1}{2} x^\top Q x + c^\top x \colon Ax \leq b, \; Cx = d \;\text{ for all } u \in \mathcal{U} \right\} \right\}$$

Some Literature

The first paper and the standard textbook

- Soyster (OR, 1973): First paper on robust (linear) optimization
- Ben-Tal, El Ghaoui, Nemirovski (2009): Seminal textbook

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- Soyster (OR, 1973): First paper on robust (linear) optimization
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Many extensions

- Bertsimas, Sim (2003, 2004), Sim (2004): Γ-robustness
- Fischetti, Monaci (2009): light robustness
- Ben-Tal, Goryashko, Guslitzer, Nemirovski (2004): adjustable robustness

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The Linear Complementarity Problem (LCP)

Given $q \in \mathbb{R}^n$, $M \in \mathbb{R}^{n \times n}$, find a vector z that satisfies

$$z \geq 0, \quad Mz + q \geq 0, \quad z^\top (Mz + q) = 0$$

or show that no such vector exists.

The Linear Complementarity Problem (LCP)

Given $q \in \mathbb{R}^n$, $M \in \mathbb{R}^{n \times n}$, find a vector z that satisfies

$$z \geq 0, \quad Mz + q \geq 0, \quad z^\top (Mz + q) = 0$$

or show that no such vector exists.

Alternative notation for the LCP(q, M)

$$0 \leq z \perp Mz + q \geq 0$$

Why Would Anyone Care?

The applications are extremely manifold!

Why Would Anyone Care?

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- Matrix theory
- Optimality conditions of QPs
- The bimatrix game is an LCP
- Market equilibrium modeling
- Optimal stopping
- Contact mechanics
- Special case of variational inequalities
- ...

Example #1

The QP

$$\min_x \quad c^\top x + \frac{1}{2} x^\top Q x \quad \text{s.t.} \quad x \geq 0$$

with positive semi-definite Q is equivalent to the LCP(q, M).

Example #1

The QP

$$\min_{x} \quad c^\top x + \frac{1}{2} x^\top Q x \quad \text{s.t.} \quad x \geq 0$$

with positive semi-definite Q is equivalent to the LCP(q, M).

- Simply write down its KKT conditions
- Can be generalized to QPs with arbitrary inequality constraints

Example #2: Market Equilibrium Modeling

Standard micro-economic setting

$$\begin{aligned} & \text{Production/Generation} \\ + \; & \text{Demand (depending on market price)} \\ + \; & \text{Market clearing conditions} \\ = \; & \text{Market equilibrium problem} \end{aligned}$$

Example #2: Market Equilibrium Modeling

Standard micro-economic setting

Production/Generation

$+$ Demand (depending on market price)

$+$ Market clearing conditions

$=$ Market equilibrium problem

$=$ LCP (under suitable assumptions)

Market Equilibrium Modeling (from Cottle, Pang, Stone 2009)

Production (modeled as an LP)

$$\begin{aligned} \min_{x \in \mathbb{R}^n} \quad & c^\top x \\ \text{s.t.} \quad & Ax \geq b \qquad [\lambda] \\ & Bx \geq r \qquad [\pi] \\ & x \geq 0 \end{aligned}$$

Demand

$$r = Dp + d$$

Equilibrating condition

$$p = \pi$$

Market Equilibrium Modeling (from Cottle, Pang, Stone 2009)

Take the production KKTs and massage the terms . . .

$$0 \le x \perp c - A^\top \lambda - B^\top p \ge 0$$
$$0 \le \lambda \perp -b + Ax \ge 0$$
$$0 \le p \perp -Dp - d + Bx \ge 0$$

This is the LCP(q, M) with

$$x = \begin{pmatrix} x \\ \lambda \\ p \end{pmatrix}, \quad M = \begin{bmatrix} 0 & -A^\top & -B^\top \\ A & 0 & 0 \\ B & 0 & -D \end{bmatrix}, \quad q = \begin{pmatrix} c \\ -b \\ -d \end{pmatrix}$$

Overview

Why Robust LCPs?

Production

$$\begin{aligned} \min_{x \in \mathbb{R}^n} \quad & c^\top x \\ \text{s.t.} \quad & Ax \geq b \\ & Bx \geq r \\ & x \geq 0 \end{aligned}$$

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Production

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Demand

$$r = Dp + d$$

Uncertainties are everywhere!

- Price sensitivity D, d
- Production data B (e.g., renewable power production)
- Cost data c (e.g., feed-in tariffs for renewables)

The Stochastic Way

Consider the LCP's gap function QP

$$\min_{x \in \mathbb{R}^n} \quad g(x) := x^\top (Mx + q)$$
$$\text{s.t.} \quad x \in \mathcal{X} := \{x \in \mathbb{R}^n \colon x \geq 0,\ Mx + q \geq 0\}$$

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Consider the LCP's gap function QP

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No-Brainer: A point $x \in \mathbb{R}^n$ is a solution of the LCP if and only if it is global minimizer of the gap function with objective function value 0.

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Expected Gap Minimization Problem

$$\min_{\text{“}x\in\mathcal{X}\text{''}} \ \mathbb{E}_{(u_M,u_q)}\left[g(x; u_M, u_q)\right]$$

with

$$g(x; u_M, u_q) := x^\top(M(u_M)x + q(u_q)).$$

Some(!) articles: Chen, Fukushima (MOR 2005),
Lin, Fukushima (OMS 2006),
Chen, Zhang, Fukushima (Math. Prog. 2009),
Chen, Wets, Zhang (SIOPT 2012)

The Robust LCP

- Consider LCP data M and q to be uncertain
- No assumptions on probability distributions
- $M(u_M)$ and $q(u_q)$ with $u_M \in \mathcal{U}_M$ and $u_q \in \mathcal{U}_q$
- $\mathcal{U}_M$, $\mathcal{U}_q$ are given (deterministic) uncertainty sets
- Example: $q(u_q) := \bar{q} + u_q$ with nominal value $\bar{q}$ and $u_q \in \mathcal{U}_q$

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The robust LCP (= family of LCPs)

$$\{0 \leq x \perp M(u_M)x + q(u_q) \geq 0\}_{(u_M, u_q) \in \mathcal{U}_M \times \mathcal{U}_q}$$

Robust Feasible Points and Robust Solutions

We call a point x strictly robust feasible if

$$x \geq 0, \quad M(u_M)x + q(u_q) \geq 0$$

holds for all $(u_M, u_q) \in \mathcal{U}_M \times \mathcal{U}_q$.

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The point is called a strictly robust LCP solution if it additionally satisfies

$$x^\top (M(u_M)x + q(u_q)) = 0 \quad \text{for all } (u_M, u_q) \in \mathcal{U}_M \times \mathcal{U}_q.$$

The Robust Gap Function Formulation

Robustifying the Gap Function QP

$$\min_{x \in \mathcal{X}(u_M, u_q)} \quad \sup_{(u_M, u_q) \in \mathcal{U}_M \times \mathcal{U}_q} \quad g(x; u_M, u_q)$$

with robust feasible set

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This is surprisingly new stuff ...

- First paper by Wu, Han, Zhu in (2011)
- Latest paper (before our articles): Xie, Shanbhag (2014, 2016)
- Nothing in between!

The Economist's Problem

Our starting point was Xie, Shanbhag (SIOPT 2016).

Proposition

A vector x solves

$$0 \leq x \perp M(u_M)x + q(u_q) \geq 0 \quad \text{for all } (u_M, u_q) \in \mathcal{U}_M \times \mathcal{U}_q$$

if and only if x is a solution of the robust gap function formulation with optimal objective function value of zero.

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Bad news

- This is almost never the case!
- "almost never" = only in trivial cases

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Bad news

- This is almost never the case!
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This means (for instance):

If the LCP models a market equilibrium,
there is no "robust market equilibrium".

Remedies?

- Let's mimic the robust optimization literature starting from 2003 on
- Thus: Consider less conservative notions of robustness
- Xie, Shanbhag (SIOPT 2016) "only" considered the strictly robust case, which delivers the most conservative solutions of all robustness concepts

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- Let's mimic the robust optimization literature starting from 2003 on
- Thus: Consider less conservative notions of robustness
- Xie, Shanbhag (SIOPT 2016) "only" considered the strictly robust case, which delivers the most conservative solutions of all robustness concepts

What we did:

- Γ-robust LCPs (à la Bertsimas, Sim (2003, 2004) and Sim (2004))
  - Krebs, S. (OMS, 2020): ℓ_1 and ℓ_∞ norm uncertainties
  - Krebs, Müller, S. (Preprint, 2019): ellipsoidal uncertainty sets
- Adjustable Robustness (à la Ben-Tal et al. (2004))
  - Biefel, Rolfes, Liers, S. (Preprint, 2020)
- Applications in power market equilibrium models
  - Kramer, Krebs, S. (Preprint, 2018)
  - Çelebi, Krebs, S. (Energy Systems, 2020)

Another Economist's Problem

If existence of robust equilibria cannot be established . . .

What about approximate equilibria?

Another Economist's Problem

If existence of robust equilibria cannot be established . . .

What about approximate equilibria?

That is, we consider solutions of

$$\min_{x \in \mathcal{X}(u_M, u_q)} \sup_{(u_M, u_q) \in \mathcal{U}_M \times \mathcal{U}_q} g(x; u_M, u_q)$$

with strictly positive optimal objective function values.

Another Economist's Problem

If existence of robust equilibria cannot be established ...

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Questions?

- Do these approximate equilibria exist?

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The Optimizer's Problem:

Another Economist's Problem

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Questions?

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- What about uniqueness?

The Optimizer's Problem:

What about tractability?

Overview

Uncertain LCP Vector q

- Consider

$$\{0 \leq x \perp Mx + q(u) \geq 0\}_{u \in \mathcal{U}}$$

- Given uncertainty set $\mathcal{U} \subset \mathbb{R}^n$
- Γ-version of the uncertainty set

$$\mathcal{U}_\Gamma := \{u \in \mathcal{U} \colon |\{i \in [n] \colon u_i \neq 0\}| \leq \Gamma\}$$

- Robust gap function problem

$$\min_x \sup_{u \in \mathcal{U}_\Gamma} \left\{x^\top Mx + x^\top q(u) \colon x \geq 0,\ Mx \geq -q(u) \text{ for all } u \in \mathcal{U}_\Gamma\right\}$$

Uncertain q . . . Bad News Again

Robust gap function problem

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Proposition (Krebs, S. (OMS, 2020))

A vector x solves

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if and only if x is a solution of the problem above with optimal objective function value of zero.

Uncertain q ... Bad News Again

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Roadmap

- Tractability of the robust gap function problem
- Existence and uniqueness of approximate robust equilibria

Uncertain q: Box Uncertainties

- $\mathcal{U}_\Gamma$: box uncertainty set

$$\mathcal{U}^{\text{box}}_{\Gamma,\bar{u}} := \{u \in \mathbb{R}^n \colon -\bar{u}_i \leq u_i \leq \bar{u}_i,\ i \in [n],\ |\{i \in [n] \colon u_i \neq 0\}| \leq \Gamma\}$$

- $\bar{u}_i \geq 0$ for all $i \in [n]$
- $q(u) := \bar{q} + u$ with $u \in \mathcal{U}^{\text{box}}_{\Gamma,\bar{u}}$

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- $q(u) := \bar{q} + u$ with $u \in \mathcal{U}_{\Gamma,\bar{u}}^{\text{box}}$

The robust counterpart of the gap function problem in this case reads

$$\begin{aligned} \min_{x \ge 0} \quad & x^\top M x + x^\top \bar{q} + \max_{\{I \subseteq [n] \colon |I| \le \Gamma\}} \sum_{i \in I} \bar{u}_i x_i \\ \text{s.t.} \quad & Mx \ge -\bar{q} + \sum_{i \in I} \bar{u}_i e_i \quad \text{for all } I \subseteq [n],\ |I| \le \Gamma \end{aligned}$$

Uncertain q: Box Uncertainties

Theorem

The robust counterpart (of the last slide) is equivalent to

$$\begin{aligned}
\min_{x,\alpha,\beta} \quad & x^\top M x + x^\top \bar{q} + \alpha\Gamma + \sum_{i=1}^{n} \beta_i \\
\text{s.t.} \quad & M_{i,\cdot} x \geq -\bar{q}_i + \bar{u}_i, \quad i \in [n] \\
& \alpha + \beta_i \geq \bar{u}_i x_i, \quad i \in [n] \\
& \alpha \geq 0 \\
& x_i \geq 0, \ \beta_i \geq 0, \quad i \in [n]
\end{aligned}$$

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& \alpha \geq 0 \\
& x_i \geq 0,\ \beta_i \geq 0, \quad i \in [n]
\end{aligned}
$$

Proof.

Read the PhD thesis of Melvyn Sim and apply the same techniques. □

Uncertain q: Box Uncertainties

Results

- The robust counterpart is convex if M is positive semi-definite
- In this case, existence of approximate robust equilibria can be shown
- If M is positive definite, the approximate robust equilibrium is unique in x
- Uniqueness of the other "primal" variables cannot be achieved

Uncertain q: Box Uncertainties

Results

- The robust counterpart is convex if M is positive semi-definite
- In this case, existence of approximate robust equilibria can be shown
- If M is positive definite, the approximate robust equilibrium is unique in x
- Uniqueness of the other "primal" variables cannot be achieved

We have similar results for the case of ℓ_1 norm uncertainties

Uncertain M: Box Uncertainties

- We now consider the problem

$$\{0 \leq x \perp M(u)x + q \geq 0\}_{u \in \mathcal{U}}$$

Uncertain M: Box Uncertainties

- We now consider the problem

$$\{0 \le x \perp M(u)x + q \ge 0\}_{u \in \mathcal{U}}$$

- Let's start with a definition of $M(u)$ in analogy to $q(u)$:

$$\bar{M} := [\bar{m}_{ij}]_{i,j \in [n]}$$

with

$$M(u) := [\bar{m}_{ij} + u_{ij}]_{i,j \in [n]}$$

and

$$[u_{ij}]_{i,j \in [n]} \in \mathcal{U}.$$

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- Box uncertainties for entries of M (row-wise)

$$\mathcal{U}^{\text{box}}_{\Gamma,\bar{u},i} := \{u_i \in \mathbb{R}^n \colon -\bar{u}_{ij} \leq u_{ij} \leq \bar{u}_{ij},\, j \in [n],\, |\{j \in [n] \colon u_{ij} \neq 0\}| \leq \Gamma_i\}$$

Uncertain M: Box Uncertainties

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- Robust counterpart

$$\begin{aligned}
\min_{x \geq 0} \quad & x^\top \bar{M} x + x^\top q + \sum_{i \in [n]} \max_{\{I_i \subseteq [n] \colon |I_i| \leq \Gamma_i\}} \sum_{j \in I_i} \bar{u}_{ij} x_i x_j \\
\text{s.t.} \quad & \sum_{j \in [n]} \bar{m}_{ij} x_j - \max_{\{I_i \subseteq [n] \colon |I_i| \leq \Gamma_i\}} \sum_{j \in I_i} \bar{u}_{ij} x_j \geq -q_i, \quad i \in [n]
\end{aligned}$$

A Bad-News-Theorem

Theorem

Let $\mathcal{U}^{box}_{\Gamma,\bar{u},i}$ be the uncertainty set of row $i \in [n]$ in $M(u)x + q \geq 0$. Then, the robust counterpart (of the last slide) is equivalent to

$$
\begin{aligned}
\min_{x,\alpha,\beta,\gamma,\delta,\varepsilon,\xi} \quad & x^\top M x + x^\top q + \sum_{i \in [n]} \left(\gamma_i \Gamma_i + \sum_{j \in [n]} \delta_{ij} \right) \\
s.t. \quad & \sum_{j \in [n]} \bar{m}_{ij} x_j - \varepsilon_i \Gamma_i - \sum_{j \in [n]} \xi_{ij} \geq -q_i, \quad i \in [n] \\
& \varepsilon_i + \xi_{ij} \geq \bar{u}_{ij} x_j, \quad j \in [n] \\
& \varepsilon_i \geq 0, \quad i \in [n] \\
& \xi_{ij} \geq 0, \quad i,j \in [n] \\
& \gamma_i + \delta_{ij} \geq \bar{u}_{ij} x_i x_j, \quad i,j \in [n] \\
& \gamma_i \geq 0, \quad i \in [n] \\
& \delta_{ij} \geq 0, \quad i,j \in [n]
\end{aligned}
$$

Sometimes Reading Helps . . . Xie, Shanbhag (2016)

The "correct" uncertainty modeling:

$$M(u) := \bar{M} + \sum_{\ell \in [L]} u_\ell M^\ell$$

with $L \in \mathbb{N}$ and $M^\ell := [m_{ij}^\ell]_{i,j \in [n]} \in \mathbb{R}^{n \times n}$

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with $L \in \mathbb{N}$ and $M^\ell := [m^\ell_{ij}]_{i,j \in [n]} \in \mathbb{R}^{n \times n}$

Uncertainty set:

$$\mathcal{U}^{\text{box}}_{\Gamma,\bar{u}} := \{u \in \mathbb{R}^L \colon 0 \leq u_\ell \leq \bar{u}_\ell,\ \ell \in [L],\ |\{\ell \in [L] \colon u_\ell \neq 0\}| \leq \Gamma\}$$

Good-News-Theorem

Theorem

Consider the "correct" uncertainty set with $L > \Gamma$. Furthermore, suppose that $\bar{M}$ and M^ℓ, $\ell \in [L]$, are positive semidefinite. Then, the robust counterpart is equivalent to the convex, and thus tractable, problem

$$
\begin{aligned}
\min_{x,\alpha,\beta,\gamma,\delta} \quad & x^\top \bar{M} x + x^\top q + \Gamma\alpha + \sum_{\ell \in [L]} \beta_\ell \\
\text{s.t.} \quad & \alpha + \beta_\ell \geq \bar{u}_\ell x^\top M^\ell x, \quad \ell \in [L] \\
& \alpha \geq 0 \\
& \beta_\ell \geq 0, \quad \ell \in [L] \\
& \gamma_i \geq 0, \quad i \in [n] \\
& \delta_{i\ell} \geq 0, \quad i \in [n],\ \ell \in [L] \\
& x \geq 0 \\
& \bar{M}_{i,\cdot} x + q_i - \gamma_i \Gamma - \sum_{\ell \in [L]} \delta_{i\ell} \geq 0, \quad i \in [n] \\
& \gamma_i + \delta_{i\ell} \geq -\bar{u}_\ell M^\ell_{i,\cdot} x, \quad i \in [n],\ \ell \in [L]
\end{aligned}
$$

Further Results

Theorem
Assume that the Problem of the last slide is feasible and that $\bar{M}$ *and* M^ℓ, $\ell \in [L]$, *are positive semidefinite. Then, there exists a solution.*

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Suppose that the matrix $\bar{M}$ is positive definite. Then, the solution of Problem is unique in x.

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Comparable results for ℓ_1 norm uncertainty sets

Uncertain q and M

No-Brainer

- If the uncertainties for q and M are independent, we can simply combine the separate robustifications.

Open problem

- Correlation between uncertainty in q and M

Ellipsoidal Uncertainties (Krebs, Müller, S. (2019))

- We obtain qualitatively comparable results
- The required techniques are a bit different
- Some of the results change as expected
  - Tractable counterparts under the assumption positive semidefinite LCP matrix . . .
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- Much harder to achieve
- Before: mainly Frank–Wolfe theorem
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A convex set $\mathcal{C} \subseteq \mathbb{R}^n$ is called a quasi-Frank-and-Wolfe set, if every quadratic function f, which is quasi-convex and bounded from below on $\mathcal{C}$, attains its infimum on $\mathcal{C}$.

Overview

Reminder: The Economist's Problem!

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that allows to prove the existence of robust equilibria?

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Good news

- Adjustable robustness works!

Adjustable Robust LCPs

Find a vector $r \in \mathbb{R}^n$, which can be adjusted for all uncertainties $(\zeta, u) \in \mathcal{U}_M \times \mathcal{U}_q$ by a vector $y(\zeta, u)$ such that $z(\zeta, u) := r + y(\zeta, u)$ satisfies

$$0 \leq z(\zeta, u) \perp M(\zeta)z(\zeta, u) + q(u) \geq 0 \quad \text{for all} \quad (\zeta, u) \in \mathcal{U}_M \times \mathcal{U}_q.$$

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Biefel, Liers, Rolfes, S. (2020)

- Box uncertainties
- Affine decision rules
- Existence, characterization, and uniqueness of solutions

Overview

Conclusion

Done

- Robust LCPs is a very young field of research
  - First two papers: 2011 and 2014/2016
  - Nothing more up to now except the papers I talked about
- Existence of robust equilibria is hard to establish ...
- ... but adjustable robustness does the job!

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To-Do

- Real-World Applications
  - Challenge: Calibration of uncertainty sets
- Other robustness concepts
  - Light robustness, distributional robustness, . . .
- Correlated uncertainties between q and M

Paper & Preprints

- **Γ-Robust Linear Complementarity Problems**
  Jointly with Vanessa Krebs
  In: Optimization Methods and Software. 2020.
- **Γ-Robust Linear Complementarity Problems with Ellipsoidal Uncertainty Sets**
  Jointly with Vanessa Krebs and Michael Müller
- **Affinely Adjustable Robust Linear Complementarity Problems**
  Jointly with Christian Biefel, Frauke Liers, and Jan Rolfes
- **Γ-Robust Electricity Market Equilibrium Models with Transmission and Generation Investments**
  Jointly with Emre Çelebi and Vanessa Krebs
  Accepted for publication (10/2020) in Energy Systems
- **Strictly and Γ-Robust Counterparts of Electricity Market Models: Perfect Competition and Nash-Cournot Equilibria**
  Jointly with Anja Kramer and Vanessa Krebs

Thanks!