Existence of Energy Market Equilibria with Convex and Nonconvex Players

Julia Grübel, Olivier Huber, Lukas Hümbs, Max Klimm, **Martin Schmidt**, Alexandra Schwartz

November 12, 2021 — Trans-Atlantic Infraday Conference

Overview

General Setting

Existence of Equilibria

Algorithm

Applications

General Setting

Market Equilibrium Problems

- Important tool to model practically relevant situations
  - Power markets
  - Gas markets
  - Auctions
  - Transport planning
  - … and many more …

Market Equilibrium Problems

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- Rationality = utility maximization
- Market should clear so that no player can improve her utility by unilaterally changing her decision

Market Equilibrium Problems

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  - … and many more …
- Rational players compete for a set of goods
- Rationality = utility maximization
- Market should clear so that no player can improve her utility by unilaterally changing her decision
- Classic questions
  - Does an equilibrium exist?
  - Is it unique?
  - How can we compute such an equilibrium?

The Convex World

Suitable convexity assumptions allow to prove the existence of market equilibria

- Wald (1951)
- Arrow, Debreu (1954)
- Gale (1955)
- McKenzie (1959)
- Debreu (1962)

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Bad news

- The world is not convex!
- Two major reasons for nonconvexity
    - Nonlinear modeling of physics
    - Mixed-integer modeling of discrete controls

We are not the first ones ...

- Assignment problems
    - Shapley and Shubik (1971), Leonard (1983), Bikhchandani, Ostroy, et al. (2002), Bikhchandani and Ostroy (2006)
- General exchange economies with indivisibilities
    - Bikhchandani and Mamer (1997), Baldwin and Klemperer (2019)
- Discrete markets
    - O'Neill et al. (2005), Guo et al. (2021)
- Trading networks
    - Hatfield et al. (2013), Fleiner et al. (2019), Hatfield et al. (2019)
- Economies with increasing returns to scale
    - Beato (1982), Brown et al. (1986), Bonnisseau and Cornet (1988), Bonnisseau and Cornet (1990)

The paper in the background

Pricing in Resource Allocation Games Based on Lagrangean Duality and Convexification

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First version, July 2019, this version February 24, 2020

Abstract

We consider a basic resource allocation game, where the players' strategy spaces are subsets of $\mathbb{R}^m$ and cost/utility functions are parameterized by some common vector $u \in \mathbb{R}^m$ and, otherwise, only depend on the own strategy choice. A strategy of a player can be interpreted as a vector of resource consumption and a joint strategy profile naturally leads to an aggregate consumption vector. Resources can be priced, that is, the game is augmented by a price vector $\lambda \in \mathbb{R}^m_+$ and players have quasi-linear overall costs/utilities meaning that in addition to the original costs/utilities, a player needs to pay the corresponding price per consumed unit. We investigate the following question: for which aggregated consumption vectors u can we find prices λ that induce an equilibrium realizing the targeted consumption profile?

For answering this question, we revisit a well-known duality-based framework and derive several characterizations of the existence of such u and λ using convexification techniques. We show that for finite strategy spaces or certain concave games, the equilibrium existence problem reduces to solving a well-structured LP. We then consider a class of monotone aggregative games having the property that the cost/utility functions of players may depend on the induced load of a strategy profile. For this class, we show a sufficient condition of enforceability based on the previous characterizations. We demonstrate that this framework can help to unify parts of four largely independent streams in the literature: tolls in transportation systems, Walrasian market equilibria, trading networks and congestion control in communication networks. Besides reproving existing results we establish new existence results by using methods from polyhedral combinatorics, polymatroid theory and discrete convexity.

The paper in the background

- Unifying framework for many (possibly nonconvex) equilibrium problems including
  - network tolls for transportation networks
  - indivisible item auctions
  - bilateral trade
  - congestion control
  - energy markets
- Framework is based on Lagrangian duality
- Enables to characterize the existence of solutions to (possibly) nonconvex equilibrium problems
- Main idea: check if a suitably chosen optimization problem has a zero duality gap

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Modeling

- Each player $i \in I$ solves

$$\min_{z_i} \quad f_i(z_i, p) = c_i(z_i) + p^\top h_i(z_i) \quad \text{s.t.} \quad z_i \in Y_i$$

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- Market-clearing condition

$$\sum_{i \in I} h_i(z_i) = 0$$

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- Putting this all together defines the market equilibrium problem
- In other words: this is a GNEP and we look for a variational equilibrium

Existence of Equilibria

Existence

- Part 1 of Theorem 2.3 in Harks (2020)
  - In the case of existence, market equilibria correspond to welfare optima
- Welfare optimization problem

$$\min_{z} \quad \sum_{i \in I} c_i(z_i) \quad \text{s.t.} \quad z \in Y, \quad \sum_{i \in I} h_i(z_i) = 0$$

- Bad news (due to nonconvexity)
  - Even if a solution to the welfare optimization problem exists, this solution does not necessarily constitute a market equilibrium if nonconvexities are present in the players' problems.

Main Ideas

Idea #1

- Market equilibria $\Longrightarrow$ primal-dual solution pairs of the welfare problem with zero duality gap

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- First compute a solution y^* of the welfare problem
- Then check if there are prices p^* so that (y^*, p^*) is a market equilibrium

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Contribution

- Assumptions under which it is enough to check a single critical price vector ...
  - to obtain a market equilibrium
  - or to prove that none can exist

Lagrangian of the welfare optimization problem

$$L(z, p) := \sum_{i \in I} \Big(c_i(z_i) + p^\top h_i(z_i) \Big)$$

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Weak duality

$$\inf_{z \in Y} \left\{ \sum_{i \in I} c_i(z_i) : z \in Y, \ \sum_{i \in I} h_i(z_i) = 0 \right\} \geq \sup_{p \in \mathbb{R}^{n_p}} d(p).$$

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Attention: duality gap of the welfare problem can be positive in the presence of nonconvexities

Theorem (see Part 1 of Theorem 2.3 in Harks (2020))

The pair (y^*, p^*) is a market equilibrium if and only if y^* and p^* are solutions of the welfare optimization problem and the corresponding dual problem, respectively, with zero duality gap.

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Immediate consequences

- If (y^*, p^*) is a market equilibrium, then y^* is a global solution of the welfare problem.
- If (y^*, p^*) is a market equilibrium, then (y, p^*) is a market equilibrium for all global solutions y of the welfare problem.
- If (y^*, p^*) and $(\hat{y}, \hat{p})$ are two market equilibria of, then so are $(y^*, \hat{p})$ and $(\hat{y}, p^*)$.
- If y^* is a global solution of the welfare problem, for which there exists no p such that (y^*, p) is a market equilibrium, then the market equilibrium problem has no solution.

Corollary

Let $S \subseteq I$ be the set of players with unique best responses for all price vectors $p \in \mathbb{R}^{n_p}$.

(a) *If (z^*, p^*) and $(\hat{z}, \hat{p})$ are two market equilibria, then $z^*_S = \hat{z}_S$.*

(b) *If z^* and $\hat{z}$ are two solutions of the welfare problem with $z^*_S \neq \hat{z}_S$, then the market equilibrium problem does not have a solution.*

Algorithm

Idea #2 (Revisited)

1. Compute a solution y^* of the welfare problem
2. Check if there are prices p^* so that (y^*, p^*) is a market equilibrium

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To determine such an equilibrium price p^*, let $\Pi(z^*) \subseteq \mathbb{R}^{n_p}$ be a set that includes all market equilibrium prices, i.e., it has the property

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Assumption

We can compute an enclosing box of $\Pi(z^*)$ which takes the form

$$\{p \in \mathbb{R}^{n_p} : p_i^- \leq p_i \leq p_i^+ \text{ for all } i \in \{1, \dots, n_p\}\}$$

Theorem

Let z^* be a solution of the welfare and let $\Pi(z^*) \neq \emptyset$ be a set satisfying the "enclosing-box condition". Assume that for all $i \in \{1, \dots, n_p\}$ at least one of the following properties is satisfied:

(a) $p_i^- = p_i^+$,

(b) $p_i^+ < \infty$ *and* $(h_i(z_i^*))_i \leq (h_i(z_i))_i$ *for all* $z_i \in Y_i$ *and all players* $i \in I$,

(c) $p_i^- > -\infty$ *and* $(h_i(z_i^*))_i \geq (h_i(z_i))_i$ *for all* $z_i \in Y_i$ *and all players* $i \in I$,

(d) $p_i^- = -\infty$, $p_i^+ = \infty$, *and* $(h_i(z_i^*))_i = (h_i(z_i))_i$ *for all* $z_i \in Y_i$ *and all players* $i \in I$.

Then, there exists a market equilibrium if and only if $(z^*, \hat{p})$ is a market equilibrium, where the critical price $\hat{p}$ is defined as

$$\hat{p}_i := \begin{cases} p_i^- = p_i^+, & \text{if (a) applies,} \\ p_i^+, & \text{if (b) applies,} \\ p_i^-, & \text{if (c) applies,} \\ 0, & \text{if (d) applies.} \end{cases}$$

A simple yet effective algorithm

Input: Market equilibrium problem

Compute a global solution z^* of the welfare optimization problem.
if *the welfare optimization problem does not have a solution* **then**
    return *"No market equilibrium exists."*
else
    Define the critical price vector $\hat{p}$ as in the last theorem.
    if z_i^* *is a best response to the price vector* $\hat{p}$ *for all players* $i \in I$ **then**
        return $(z^*, \hat{p})$ *is a market equilibrium.*
    else
        return *"No market equilibrium exists."*
    end
end

Applications

Network

- Network is modeled as a directed and weakly connected graph $G = (V, A)$
- $N = N_{\mathrm{d}} \cup N_s \cup N_0$
  - $N_{\mathrm{d}} \subset N$: consumer locations
  - $N_s \subset N$: producer locations
  - $N_0 \subset N$: inner nodes

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    - $N_0 \subset N$: inner nodes

Players

- Consumers

$$\max_{d_n} \quad \int_0^{d_n} P_n(\omega)\,\mathrm{d}\omega - p_n d_n \quad \text{s.t.} \quad d_n \geq 0.$$

- Producers

$$\max_{y_n} \quad p_n y_n - c_n(y_n) \quad \text{s.t.} \quad \bar{y}_n \geq y_n \geq 0.$$

Further Player: TSO

$$
\begin{aligned}
\max_{q,x} \quad & \sum_{n \in N_\mathrm{d} \cup N_\mathrm{s}} p_n \left(\sum_{a \in \delta^{\mathrm{in}}(n)} q_a - \sum_{a \in \delta^{\mathrm{out}}(n)} q_a \right) - c^{\mathrm{t}}(q,x) \\
\text{s.t.} \quad & \sum_{a \in \delta^{\mathrm{in}}(n)} q_a - \sum_{a \in \delta^{\mathrm{out}}(n)} q_a \geq 0 \quad \text{for all } n \in N_\mathrm{d} \\
& \sum_{a \in \delta^{\mathrm{in}}(n)} q_a - \sum_{a \in \delta^{\mathrm{out}}(n)} q_a \leq 0 \quad \text{for all } n \in N_\mathrm{s} \\
& \sum_{a \in \delta^{\mathrm{in}}(n)} q_a - \sum_{a \in \delta^{\mathrm{out}}(n)} q_a \geq -\bar{y}_n \quad \text{for all } n \in N_\mathrm{s} \\
& \sum_{a \in \delta^{\mathrm{in}}(n)} q_a - \sum_{a \in \delta^{\mathrm{out}}(n)} q_a = 0 \quad \text{for all } n \in N_0 \\
& F(q,x) \geq 0
\end{aligned}
$$

Market Clearing

$$\sum_{a\in\delta^{\text{in}}(n)} q_a - \sum_{a\in\delta^{\text{out}}(n)} q_a = d_n \quad \text{for all } n \in N_{\text{d}}$$

$$\sum_{a\in\delta^{\text{in}}(n)} q_a - \sum_{a\in\delta^{\text{out}}(n)} q_a = -y_n \quad \text{for all } n \in N_{\text{s}}$$

Welfare Optimization Problem

$$
\begin{aligned}
\max_{d,y,q,x} \quad & \sum_{n \in N_\mathrm{d}} \int_0^{d_n} P_n(\omega)\,\mathrm{d}\omega - \sum_{n \in N_\mathrm{s}} c_n(y_n) - c^\mathrm{t}(q,x) \\
\text{s.t.} \quad & \sum_{a \in \delta^{\mathrm{in}}(n)} q_a - \sum_{a \in \delta^{\mathrm{out}}(n)} q_a = 0 \quad \text{for all } n \in N_0 \\
& \sum_{a \in \delta^{\mathrm{in}}(n)} q_a - \sum_{a \in \delta^{\mathrm{out}}(n)} q_a = d_n \quad \text{for all } n \in N_\mathrm{d} \\
& \sum_{a \in \delta^{\mathrm{in}}(n)} q_a - \sum_{a \in \delta^{\mathrm{out}}(n)} q_a = -y_n \quad \text{for all } n \in N_\mathrm{s} \\
& F(q,x) \geq 0, \quad d \geq 0, \quad \bar{y} \geq y \geq 0
\end{aligned}
$$

Assumption

1. The inverse demand functions $P_n(\cdot)$ are continuous and strictly decreasing for all $n \in N_\mathrm{d}$
2. The variable cost functions $c_n(\cdot)$ are monotonically increasing with $c_n(0) = 0$, convex, and continuously differentiable for all $n \in N_s$

Theorem

Suppose the assumption holds. Let (d^*, y^*, q^*, x^*) be a solution of the welfare problem and define $\hat{p}$ as

$$\hat{p}_n := \begin{cases} P_n(d_n^*), & \text{if } n \in N_d, \\ c_n'(y_n^*), & \text{if } n \in N_s. \end{cases}$$

Then, either $(d^*, y^*, q^*, x^*, \hat{p})$ is a market equilibrium, or there is no market equilibrium.

Physical and technical model

$$
\begin{aligned}
p_n^2 - p_m^2 &= \Lambda_a q_a |q_a|, \quad a = (n, m) \in A \\
p_n^- \leq p_n &\leq p_n^+, \quad n \in N \\
q_a^- \leq q_a &\leq q_a^+, \quad a \in A
\end{aligned}
$$

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Transportation costs

$$
c^{\mathrm{t}}(q) = \sum_{a \in A} \alpha q_a^2
$$

Application #1: Gas Flow

Physical and technical model

$$p_n^2 - p_m^2 = \Lambda_a q_a |q_a|, \quad a = (n, m) \in A$$
$$p_n^- \leq p_n \leq p_n^+, \quad n \in N$$
$$q_a^- \leq q_a \leq q_a^+, \quad a \in A$$

Transportation costs

$$c^{\mathrm{t}}(q) = \sum_{a \in A} \alpha q_a^2$$

Bad news: There are instances that have no equilibria! (Grimm, Grübel, Schewe, S, Zöttl, 2019)

Application #1: DC Line Switching

Physical and technical model

$$
\begin{aligned}
q_a^- \le q_a \le q_a^+, &\quad a \in A_- \\
\theta_n - \theta_m - \theta_a^{\text{shift}} = B_a q_a, &\quad a = (n,m) \in A_- \\
M_a^-(1 - z_a) \le \theta_n - \theta_m - \theta_a^{\text{shift}} - B_a q_a \le M_a^+(1 - z_a), &\quad a = (n,m) \in A_+ \\
q_a^- z_a \le q_a \le q_a^+ z_a, &\quad a \in A_+ \\
z_a \in \{0, 1\}, &\quad a \in A_+
\end{aligned}
$$

Transportation costs

$$
c^{\text{t}}(q, z) = \sum_{a \in A} \alpha q_a^2 + \sum_{a \in A_+} \beta z_a
$$

Bad news: There are instances that have no equilibria!

Computational Setup

- Python 3.8.5
- Pyomo 5.7.3 (Hart et al., 2017)
- NLP solver for gas flow application: ANTIGONE 1.1 (Misener and Floudas, 2014)
- MILP solver for DC application: Gurobi 9.1.1
- Xeon E3-1240 v5 CPU (4 cores) at 3.50 GHz with 32 GB RAM

Computational Results: Gas Flow Application

- GasLib instances (`http://gaslib.zib.de`)
- Modifications as in Heitsch, Henrion, Kleinert, S. (2021) and Schewe, Thürauf, S. (2020)
- No fixed transportation costs; transportation cost factor α in $\{0.01, 0.05, 0.1\}$

Name	$\lvert N \rvert$	$\lvert N_d \rvert$	$\lvert N_s \rvert$	$\lvert A \rvert$	# instances
Gas-134-S	134	45	3	133	60
Gas-11-H	11	3	3	10	36

Results

- 84 instances are solved in 1 hour
- Average runtime: 53.1 s
- Median runtime: 18.7 s
- 56 instances have a “congested” flow situation
- We find an equilibrium for all instances

Computational Results: DC Line Switching

- MATPOWER 7.0 instances with polynomial cost functions
- Exclude instances for which minimum & maximum phase angle difference coincide in all nodes
- In total: 29 instances
- All instances with more than 1000 nodes and more than 1500 arcs cannot be solved to optimality
- 17 remaining instances
- $\alpha \in \{0.01, 0.05, 0.1\}$ and $\beta \in \{20, 50\}$
- In total: 102 instances

Instance	$\lvert N \rvert$	$\lvert N_d \rvert$	$\lvert N_s \rvert$	$\lvert A \rvert$	$\lvert A_+ \rvert$
Smallest	5	3	4	6	1
Average	56	33	15	97	10
Biggest	300	191	69	411	42

Results

- Average runtime: 0.4 s — Median runtime: 0.3 s
- 60 out of 102 instances possess an equilibrium

Open Questions & Future Work

SCAN ME

Julia Grübel, Olivier Huber, Lukas Hümbs, Max Klimm, Martin Schmidt, Alexandra Schwartz

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Open Questions

- Explain the results: What is the "real difference" between gas and power?
- What can we do if no market equilibrium exists?